PRINCIPLES AND TECHNIQUES IN COMBINATORICS

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Preface

Over the years, Combinatorial Analysis has always been a popular course among our undergraduate students. The basic principles and techniques taught in the course have found more and more applications in other fields, especially in computer science and operational research. Problems in Combinatorics are not only challenging for researchers, but also appear very frequently in various mathematical competitions, particularly the International Mathematical Olympiad (IMO). Both the authors have been involved in the teaching of the subject as well as in the training of the Singapore International Mathematical Olympiad Teams for many years. All along, we have been longing for a book that is suitable for our purposes. Hence, in writing this book, we have two main objectives in mind: (1) it could be used as a text-book for undergraduate courses, and (2) it could be used for the training of our International Mathematical Olympiad Teams. To achieve these objectives, we have tried to present the material very explicitly so that students with some mathematical maturity will find it very easy to read. We also find that students often neglect some of the basic principles in combinatorics, such as the Addition Principle, the Multiplication Principle, the Principle of Complementation, the Injection Principle, and the Bijection Principle, perhaps due to their rather unsophisticated appearances. In this book, we shall lay special emphasis on the importance of these principles, together with others, such as the Principle of Inclusion and Exclusion and the Pigeonhole Principle. By providing a plethora of carefully chosen examples, we hope that the applications of these principles as well as the techniques of generating functions and recurrence relations would be much more appreciated by the reader. We have also included a wide range of examples and exercises with various degrees of difficulty at the end of each chapter. All in all, we have about 490 problems, which include combinatorial problems taken from mathematical competitions such as the IMO, the William Lowell Putnam Mathematical Competition, the American Invitational Mathematics Examination, the Singapore Mathematical

Olympiad, the Asian Pacific Mathematics Olympiad, the USA Mathematical Olympiad and other Mathematical Olympiads from various nations. Some of the problems are specially drawn from sources, such as the American Mathematical Monthly, Crux Mathematicorum, the College Mathematics Journal and the Mathematical Magazine. We shall like to express here our gratitude to all the above publications and organizations associated with the various mathematical competitions for kindly allowing us to include these problems as exercises. The sources of these problems are clearly indicated at the end of each respective problem, so that the interested reader may consult the relevant literature for further readings. We also made an effort to be sure that the results included in here are up-to-date and to provide the reader with a good list of references at the end of each chapter and also at the end of the book so that the reader may pursue a topic further to get to the frontier of the subject.

To make reading a little easier and pleasant, a mark ∎ is placed at the end of a proof, or an example, or a solution to indicate completion. The numberings of the sections, identities, problems, figures, and tables are split into parts separated by decimal points. For instance, Problem 3.5 means the Problem 5 in Exercise 3, Section 4.2 means the second section of Chapter 4, Figure 2.3.1 means the first figure in the third section of Chapter 2, etc.. There are two kinds of references in the book. References indicated by letters in square-brackets, such as [K] or [Ro], are articles that can be found at the end of the corresponding chapter, whereas, references indicated by numbers in square-brackets, such as [3], are books that can be found in the Bibliography at the end of the book.

We wish to express our special thanks to Chan Onn for very patiently and carefully reading through the first draft of the book and for his many invaluable suggestions which certainly enhanced the contents as well as the presentation of the book. We wish to thank also our students Chan Hock Peng, Goh Beng Huay, Ng Wee Leng, Ngan Ngiap Teng, Tan Ban Pin and Teo Chung Piaw for reading through the many problems in the exercises. Last but not least, we are grateful to the National University of Singapore for granting us local leave during which this book was written.

Chen, Chuan-Chong and *Koh, Khee-Meng*

Notation and Abbreviation

$\mathbf{N}$	=	$\{1, 2, 3, \ldots\}$, (p.5)
$\mathbf{N}^*$	=	$\{0, 1, 2, 3, \ldots\}$, (p.104)
$\mathbf{N}_k$	=	$\{1, 2, 3, \ldots, k\}$, (p.62)
$\mathbf{N}_k^*$	=	$\{0, 1, 2, 3, \ldots, k\}$, (p.91)
$\mathbf{Z}$	=	$\{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$, (p.3)
$\mathbf{R}$	=	the set of real numbers, (p.187)
(AP)	:	Addition Principle, (p.1)
(BP)	:	Bijection Principle, (p.27)
(CP)	:	Complementation Principle, (p.16)
(IP)	:	Injection Principle, (p.27)
(MP)	:	Multiplication Principle, (p.4)
(PP)	:	Pigeonhole Principle, (p.120)
(GPP)	:	Generalized Pigeonhole Principle, (p.133)
(PIE)	:	Principle of Inclusion and Exclusion, (p.146)
(GPIE)	:	Generalized Principle of Inclusion and Exclusion, (p.150)
(RP)	:	Reflection Principle, (p.91)
LHS	:	Left hand side, (p.192)
RHS	:	Right hand side, (p.151)
$\Leftrightarrow$	:	if and only if, (p.79)
iff	:	if and only if, (p.95)

$a \mid b$	:	a divides b, (p.94)
$a \nmid b$	:	a does not divide b, (p.94)
$\lfloor x \rfloor$	=	the largest integer less than or equal to x (p.94)
$\lceil x \rceil$	=	the smallest integer greater than or equal to x (p.124)
$a \equiv b \pmod{m}$	:	a is congruent to b modulo m, i.e. $m \mid (a-b)$ (p.94)
HCF	:	highest common factor, (p.113)
LCM	:	Lowest common multiple, (p.147)
$\lvert S \rvert$	=	the number of elements in the finite set S, (p.2)
$s(r,n)$	=	Stirling number of the first kind
	=	the number of ways to arrange r distinct objects around n identical circles such that each circle has at least one object, (p.25)
$S(r,n)$	=	Stirling number of the second kind
	=	the number of ways of distributing r distinct objects into n identical boxes such that no box is empty, (p.47)
B_r	=	the rth Bell number $= \sum_{n=1}^{r} S(r,n)$, (p.50)
$C_r^n = \binom{n}{r}$	=	the number of r-element subsets of an n-element set
	=	$\frac{n!}{r!(n-r)!}$, (p.17)
P_r^n	=	the number of r-permutations of n distinct objects
	=	$\frac{n!}{(n-r)!}$, (p.7)
H_r^n	=	$\binom{r+n-1}{r}$, (p.37)
$P(r; r_1, \ldots, r_n)$	=	$\frac{r!}{r_1! r_2! \cdots r_n!}$, (p.34)

Q_r^n	$=$	the number of r-circular permutations of n distinct objects
	$=$	$\frac{P_r^n}{r}$, (p.13)
D_n	$=$	the number of derangements of $\mathbf{N}_n$, (p.160)
$D(n,r,k)$	$=$	the number of r-permutations of $\mathbf{N}_n$ that have exactly k fixed points, (p.160)
$\mathcal{P}(X)$	$=$	the power set of X, (p.28)
$\varphi(n)$	$=$	the Euler φ-function, (p.160)
$R(p,q)$	$=$	the smallest natural number "n" such that for any colouring of the edges of an n-clique by 2 colours: blue or red (one colour for each edge), there exists either a "blue p-clique" or a "red q-clique", (p.132)
$R(p_1,\ldots,p_n)$	$=$	the smallest natural number "n" such that for any colouring of the edges of an n-clique by k colours: colour 1, colour 2, ..., colour k, there exist a colour i ($i = 1, 2, ..., k$) and a p_i-clique in the resulting configuration such that all edges in the p_i-clique are coloured by colour i, (p.136)
$\binom{n}{n_1, n_2, ..., n_m}$	$=$	$\frac{n!}{n_1!n_2!\cdots n_m!}$, (p.96)
$p(n)$	$=$	the number of different partitions of n, (p.196)
MO	:	Mathematical Olympiad
IMO	:	International Mathematical Olympiad
APMO	:	Asian Pacific Mathematics Olympiad
AIME	:	American Invitational Mathematics Examination

Contents

Chapter 1

Permutations and Combinations

1.1. Two Basic Counting Principles

In our everyday lives, we often need to enumerate "events" such as, the arrangement of objects in a certain way, the partition of things under a certain condition, the distribution of items according to a certain specification, and so on. For instance, we may come across counting problems of the following types:

"How many ways are there to arrange 5 boys and 3 girls in a row so that no two girls are adjacent?"

"How many ways are there to divide a group of 10 people into three groups consisting of 4, 3 and 2 people respectively, with 1 person rejected?"

These are two very simple examples of counting problems related to what we call "permutations" and "combinations". Before we introduce in the next three sections what permutations and combinations are, we state in this section two principles that are fundamental in all kinds of counting problems.

The Addition Principle (AP) Assume that there are

$$\begin{array}{llll} n_1 & \text{ways for the event} & E_1 & \text{to occur,} \\ n_2 & \text{ways for the event} & E_2 & \text{to occur,} \\ & \vdots & & \\ n_k & \text{ways for the event} & E_k & \text{to occur,} \end{array}$$

where $k \geq 1$. If these ways for the different events to occur are pairwise disjoint, then the number of ways for at least one of the events $E_1, E_2, \ldots,$ or E_k to occur is $n_1 + n_2 + \cdots + n_k = \sum_{i=1}^{k} n_i$.

Example 1.1.1. One can reach city Q from city P by sea, air and road. Suppose that there are 2 ways by sea, 3 ways by air and 2 ways by road (see Figure 1.1.1). Then by (AP), the total number of ways from P to Q by sea, air or road is $2+3+2=7$. ∎

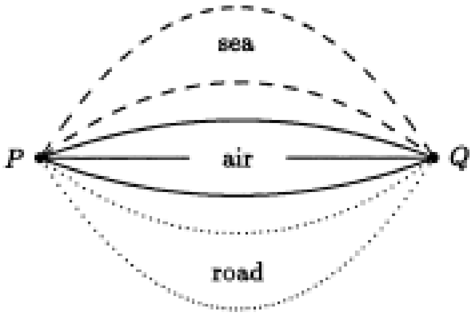

Figure 1.1.1.

An equivalent form of (AP), using set-theoretic terminology, is given below.

> Let $A_1, A_2, \ldots, A_k$ be any k finite sets, where $k \geq 1$. If the given sets are pairwise disjoint, i.e., $A_i \cap A_j = \emptyset$ for $i, j = 1, 2, \ldots, k$, $i \neq j$, then
>
> $$\left|\bigcup_{i=1}^{k} A_i\right| = |A_1 \cup A_2 \cup \cdots \cup A_k| = \sum_{i=1}^{k} |A_i|.$$

Example 1.1.2. Find the number of ordered pairs (x, y) of integers such that $x^2 + y^2 \leq 5$.

Solution. We may divide the problem into 6 disjoint cases: $x^2 + y^2 = 0, 1, \ldots, 5$. Thus for $i = 0, 1, \ldots, 5$, let

$$S_i = \{(x, y) \mid x, y \in \mathbf{Z},\ \ x^2 + y^2 = i\}.$$

It can be checked that

$S_0 = \{(0,0)\}$,

$S_1 = \{(1,0), (-1,0), (0,1), (0,-1)\}$,

$S_2 = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$,

$S_3 = \emptyset$,

$S_4 = \{(0,2), (0,-2), (2,0), (-2,0)\}$, and

$S_5 = \{(1,2), (1,-2), (2,1), (2,-1), (-1,2), (-1,-2), (-2,1), (-2,-1)\}$.

Thus by (AP), the desired number of ordered pairs is

$$\sum_{i=0}^{5} |S_i| = 1 + 4 + 4 + 0 + 4 + 8 = 21. \quad \blacksquare$$

Remarks. 1) In the above example, one can find out the answer "21" simply by listing all the required ordered pairs (x, y). The above method, however, provides us with a systematical way to obtain the answer.

2) One may also divide the above problem into disjoint cases: $x^2 = 0, 1, \ldots, 5$, find out the number of required ordered pairs in each case, and obtain the desired answer by applying (AP).

The Multiplication Principle (MP) Assume that an event E can be decomposed into r ordered events $E_1, E_2, \ldots, E_r$, and that there are

n_1 ways for the event E_1 to occur,
n_2 ways for the event E_2 to occur,
$\vdots$
n_r ways for the event E_r to occur.

Then the total number of ways for the event E to occur is given by:

$$n_1 \times n_2 \times \cdots \times n_r = \prod_{i=1}^{r} n_i.$$

Example 1.1.3. To reach city D from city A, one has to pass through city B and then city C as shown in Figure 1.1.2.

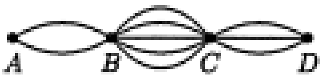

Figure 1.1.2.

If there are 2 ways to travel from A to B, 5 ways from B to C, and 3 ways from C to D, then by (MP), the number of ways from A to D via B and C is given by $2 \times 5 \times 3 = 30$. ∎

An equivalent form of (MP) using set-theoretic terminology, is stated below.

Let

$$\prod_{i=1}^{r} A_i = A_1 \times A_2 \times \cdots \times A_r = \{(a_1, a_2, ..., a_r) \mid a_i \in A_i, i = 1, 2, ..., r\}$$

denote the cartesian product of the finite sets $A_1, A_2, ..., A_r$. Then

$$\left|\prod_{i=1}^{r} A_i\right| = |A_1| \times |A_2| \times \cdots \times |A_r| = \prod_{i=1}^{r} |A_i|.$$

A sequence of numbers $a_1a_2\ldots a_n$ is called a *k-ary sequence*, where $n, k \in \mathbf{N}$, if $a_i \in \{0, 1, ..., k-1\}$ for each $i = 1, 2, ..., n$. The *length* of the sequence $a_1a_2\ldots a_n$ is defined to be n, which is the number of terms contained in the sequence. At times, such a sequence may be denoted by $(a_1, a_2, ..., a_n)$. A k-ary sequence is also called a *binary*, *ternary*, or *quaternary* sequence when $k = 2, 3$ or 4, respectively. Thus, $\{000, 001, 010, 100, 011, 101, 110, 111\}$ is the set of all $8 (= 2^3)$ binary sequences of length 3. For given $k, n \in \mathbf{N}$, how many different k-ary sequences of length n can we form? This will be discussed in the following example. You will find the result useful later on.

Example 1.1.4. To form a k-ary sequence $a_1a_2...a_n$ of length n, we first select an a_1 from the set $B = \{0, 1, ..., k-1\}$; then an a_2 from the same set B; and so on until finally an a_n again from B. Since there are k choices in each step, the number of distinct k-ary sequences of length n is, by (MP), $\underbrace{k \times k \times \cdots \times k}_{n} = k^n$. ∎

Example 1.1.5. Find the number of positive divisors of 600, inclusive of 1 and 600 itself.

Solution. We first note that the number '600' has a unique prime factorization, namely, $600 = 2^3 \times 3^1 \times 5^2$. It thus follows that a positive integer m is a divisor of 600 if and only if m is of the form $m = 2^a \times 3^b \times 5^c$, where $a, b, c \in \mathbf{Z}$ such that $0 \le a \le 3$, $0 \le b \le 1$ and $0 \le c \le 2$. Accordingly, the number of positive divisors of '600' is the number of ways to form the triples (a, b, c) where $a \in \{0, 1, 2, 3\}$, $b \in \{0, 1\}$ and $c \in \{0, 1, 2\}$, which by (MP), is equal to $4 \times 2 \times 3 = 24$. ∎

Remark. By applying (MP) in a similar way, one obtains the following general result.

If a natural number n has as its prime factorization,

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

where the p_i's are distinct primes and the k_i's are positive integers, then the number of positive divisors of n is given by $\prod_{i=1}^{r}(k_i + 1)$.

In the above examples, we have seen how (AP) and (MP) were separately used to solve some counting problems. Very often, solving a more complicated problem may require a 'joint' application of both (AP) and (MP). To illustrate this, we give the following example.

Example 1.1.6. Let $X = \{1, 2, ..., 100\}$ and let

$$S = \{(a, b, c) \mid a, b, c \in X, a < b \text{ and } a < c\}.$$

Find $|S|$.